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# OPTIMAL ONLINE DETECTION OF PARAMETER CHANGES IN TWO LINEAR MODELS

H.J. VAMAN
*Bangalore University, Bangalore 560 056, India*

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Shiryaev has obtained the optimal sequential rule for detecting the instant of a distributional change in an independent sequence using the theory of optimal stopping of Markov processes. This paper considers the problem of sequential detection of certain parameter changes in two dependent sequences: an autoregressive process, and a regression model with serially correlated error terms. It is shown that the rule that is optimal in the sense of minimizing the expected positive delay is the one which declares a change to have occurred as soon as the posterior probability of a change crosses a threshold. This rule also permits control of the probability of a false-declaration of change, just as in the independent sequence case.

disorder problem * sequential detection * autoregressive process * regression model * optimal stopping * change point

## 1. Introduction

Suppose that, during the time a sequence of random variables (r.v.'s) is being observed, the probability distribution governing it undergoes a change at some unknown instant. The problem of online detection of the epoch of change, or disorder as it is sometimes called, consists of obtaining a sequential rule which minimizes a loss associated with the delay in detection.

Let $(X_n\colon n\in\mathbb{N})$, $\mathbb{N}=\{0,1,2,\cdots\}$, defined on $(\Omega,\mathcal{A})$ be a potentially observable sequence of r.v.'s. Assume that the epoch of distributional change is a $\mathbb{N}$-valued $\mathcal{A}$-measurable r.v. $\theta$. Let $P_0$ be the probability measure governing $(X_n)$ prior to the instant of change, and $P_1$ the one *from* the change point onwards. Let $(\mathcal{F}_n)$ denote the sequence of histories of the process $(x_n)$. A sequential detection rule $\tau$ is a Markov time relative to $(\mathcal{F}_n)$; $\mathcal{T}$ be the class of $\mathcal{F}_n$-times. Shiryaev (1963) characterizes the optimal $\tau$, for a loss function to be described presently, assuming that $X_n$'s are independent and that $\theta$ has a geometric prior. Specifically, if we fix numbers $\pi$ and $p$ in $[0,1]$, they are assumed to determine a mixture $P^\pi$ which specifies the prior distribution of $\theta$ as

$$P^\pi(\theta=0)=\pi,\qquad P^\pi(\theta=j\,|\,\theta>0)=pq^{j-1},\quad j=1,2,\ldots,\tag{1}$$

where $q = 1 - p$. More on this formulation is found in Shiryaev (1978). The $\mathscr{F}_n$-time

$$\inf\{n \geq 0: \Pi_n \geq \nu\}, \quad 0 < \nu < 1, \tag{2}$$

where

$$\Pi_n = P^{\pi}(\theta \leq n \,|\, \mathscr{F}_n), \quad n \in \mathbb{N}, \tag{3}$$

has been shown by him to minimize $E^{\pi}(\tau - \theta)^+$, the expected positive detection delay, in an appropriate subclass of $\mathscr{T}$. Here $E^{\pi}$ denotes expectation under $P^{\pi}$. Similar results were arrived at for detection of disorder in a Poisson process by Davis (1974) who formulated it as a stochastic filtering problem. Bojdecki (1979) considered the problem in an independent sequence but with a new loss function. Darkhouskii and Brodskii (1980) have addressed themselves to the problem for a dependent sequence but with a fixed sample version of the problem.

The present paper shows that the optimal sequential detection rule, in the sense to be described in detail in Section 2, in the case of two dependent sequences continues to be of the type given by (2). In Section 3, we consider a possible change in the mean of an autoregressive process, and in Section 4, a possible change in the regression coefficient when the error terms in the model are serially correlated. In each case the optimal rule permits restriction of the expected number of false signals of change.

## 2. The loss function

Given a $\tau \in \mathscr{T}$, we define $\alpha(\tau) = P^{\pi}(\tau < \theta)$. Then $\alpha(\tau)$ represents the $P^{\pi}$-probability of a false signal, the event $\{\tau < \theta\}$, under the detection rule $\tau$. For each of the models in Sections 3 and 4, we shall first determine the rule $\tau^*$ such that

$$w\alpha(\tau^*) + E^{\pi}(\tau^* - \theta)^+ = \inf_{\mathscr{T}}\{w\alpha(\tau) + E^{\pi}(\tau - \theta)^+\} \tag{4}$$

for a given $w > 0$; $w$ weighs the importance of $\alpha(\cdot)$ relative to the expected delay. We next consider characterization of $\tau_\alpha^*$ satisfying

$$E^{\pi}(\tau_\alpha^* - \theta)^+ = \inf_{\mathscr{T}(\alpha)} E^{\pi}(\tau - \theta)^+ \tag{5}$$

where $\mathscr{T}(\alpha) = \{\tau \in \mathscr{T}: \alpha(\tau) \leq \alpha\}$, the class of rules with their false-signal probability not exceeding a specified number $\alpha$. Henceforth, we shall specify a stopping problem, such as the ones in (4) and (5), by stating just the right-hand side.

It is shown in Shiryaev (1963) that the problem specified by (5) is equivalent to

$$\inf_{\mathscr{T}(N)} E^{\pi}(\tau - \theta)^+ \tag{6}$$

where $\mathscr{T}(N) = \{\tau \in \mathscr{T}: N(\tau) \leq N\}$, $N(\tau)$ being the expected number of false-alarms under $\tau$, and $N$ is a predetermined bound on it. To this result we may add another equivalent version,

$$\inf_{\mathscr{T}(k)} E^{\pi}(\tau-\theta)^{+}, \tag{7}$$

where $\mathscr{T}(k)=\{\tau\in\mathscr{T}\colon E^{\pi}(\tau-\theta)^{-}\leqslant k\}$, $k$ being predetermined.

$$\begin{aligned}
E^{\pi}(\tau-\theta)^{-} &= E^{\pi}\left[\sum_{n=0}^{\infty} 1(\tau=n)\sum_{k=n}^{\infty} P^{\pi}(\theta>k\,|\,\mathscr{F}_n)\right]\\
&= E^{\pi}\left[\sum_{n=0}^{\infty} 1(\tau=n)\sum_{k=n}^{\infty}\left\{(1-\Pi_n)\sum_{j=k+1}^{\infty}\frac{P(\theta=j)}{P(\theta>n)}\right\}\right]\\
&= E^{\pi}\left[(1-\Pi_{\tau})\sum_{k=\tau}^{\infty}\frac{P(\theta>k)}{P(\theta>\tau)}\right],
\end{aligned}$$

where $1(A)$ denotes the indicator function of $A$. When $\theta$ has the geometric prior given by (1),

$$E^{\pi}(\tau-\theta)^{-}=E^{\pi}\frac{(1-\Pi_{\tau})}{p}=\frac{\alpha(\tau)}{p}.$$

This establishes the equivalence of the problems in (5) and (7). The version in (7) is analogous to the approach which seeks the rule that minimizes $\mathrm{ARL}_1$ (defined as the expected value of $\tau$ under $P_1$) in the class of procedures with $\mathrm{ARL}_0$ (defined as the expected value of $\tau$ under $P_0$) at least as large as some specified number. Khan (1979) considers this for an independent sequence and suggests a cusum type procedure.

## 3. Detecting a shift in the mean of an autoregressive process

### 3.1. The model

Let the process $(X_n)$ be a normal Markov sequence with $X_0=0$, and let $\theta$ have the prior distribution stated in (1). Then,

$$\begin{aligned}
\text{for } k=0,1,\quad & X_1=\mu+\delta+Y_1,\\
& X_n=\mu+\delta+\rho(X_{n-1}-\mu-\delta)+Y_n \quad (n=2,3,\cdots);\\
\text{for } k>1,\quad & X_n=\mu+\rho(X_{n-1}-\mu)+Y_n \quad (n=1,2,\ldots,k-1),\\
& \quad\;\, =\mu+\delta+\rho(X_{n-1}-\mu)+Y_n \quad (n=k),\\
& \quad\;\, =\mu+\delta+\rho(X_{n-1}-\mu-\delta)+Y_n \quad (n=k+1,\cdots),
\end{aligned} \tag{8}$$

where $(i)$ $Y_n\sim\mathscr{N}_1(0,1)$ $(n\in\mathbb{N})$ are independent $(ii)$ $|\rho|<1$, and $-\infty<\mu,\ \delta<\infty$. The model assumes that disorder results from a shift of magnitude $\delta$ (known) in the mean of an autoregressive process of order 1, i.e., given $\theta=k$, $EX_n=\mu$ for $n<k$, and $\mu+\delta$ for $n\geqslant k$. The parameters $\mu$ and $\rho$ are assumed known.

We transform $(X_n)$ to $(Z_n)$ by taking

$$Z_1 = X_1, \qquad Z_n = X_n - \rho X_{n-1} \quad (n = 2, 3, \ldots).$$

Let $Z(n|k)$ denote the sample $(Z_1, Z_2, \ldots, Z_n)$ when $\theta = k$. Then, we can show that

$$Z(n|k) \sim \mathcal{N}_n(\mu(n|k), I(n)), \tag{9}$$

i.e., it has a multinormal distribution with mean vector

$$\begin{aligned} \mu(n|k) &= (\mu, \bar{\rho}\mu, \ldots, \bar{\rho}\mu, \bar{\rho}\mu + \delta, \bar{\rho}(\mu+\delta), \ldots, \bar{\rho}(\mu+\delta)) \quad (k > 1), \\ &= (\mu + \delta, \bar{\rho}(\mu+\delta), \ldots, \bar{\rho}(\mu+\delta)) \qquad (k = 0 \text{ or } 1) \end{aligned} \tag{10}$$

and dispersion matrix $I(n)$, the $n \times n$ identity matrix. In (10), the element $\bar{\rho}\mu + \delta$ is the k-th one among $n$, $\bar{\rho} = 1 - \rho$. Evidently $Z_1, Z_2, \ldots, Z_n$ are mutually independent for every $n$, and each is normal with a mean that depends upon the value of $\theta$.

### 3.2. The posterior process

Given $X(n) = x(n)$, we can compute $z(n)$ and use Bayes formula to get $\pi_n$. Let $f(z(n)|k)$ stand for the joint density of $Z(n|k)$. Then, for $1 \leq k \leq n$,

$$P^{\pi}(\theta = k | \mathcal{F}_n) = \frac{(1-\pi)pq^{k-1}f(z(n)|k)}{\pi f(z(n)|0) + (1-\pi)\sum_{i=1}^{n} pq^{i-1}f(z(n)|i) + \sum_{j=n+1}^{\infty} pq^{j-1}f(z(n)|\theta > n)},$$

and when $k = 0$, $P^{\pi}(\theta = 0|\mathcal{F}_n)$ is proportional to $\pi f(z(n)|0)$. Notice that $f(z(n)|0) = f(z(n)|1)$, $\forall n \geq 1$. Now, set

$$\begin{aligned} y_k^n &= q^{k-n-1} \cdot \frac{f(z(n)|k)}{f(z(n)|\theta > n)} \quad (1 \leq k \leq n), \\ &= 1 \qquad (k > n). \end{aligned} \tag{11}$$

In terms of $y_k^n$'s,

$$\begin{aligned} p^{\pi}(\theta = k|\mathcal{F}_n) &= \frac{py_k^n}{1 + \frac{\pi}{1-\pi}y_1^n + p\sum_{j=1}^{n} y_j^n} \quad (1 \leq k < n), \\ &= \frac{pq^{k-n-1}}{1 + \frac{\pi}{1-\pi}y_1^n + p\sum_{j=1}^{n} y_j^n} \quad (k \geq n), \end{aligned}$$

and on summing,

$$1 - \pi_n = \left(1 + \frac{\pi}{1-\pi}y_1^n + p\sum_{j=1}^{n} y_j^n\right)^{-1} \quad (n \geq 1). \tag{12}$$

**Lemma 1.** *The process $(\Pi_n)_0^\infty$ is Markov relative to $(\mathscr{F}_n)$, and further $(\Pi_n)_2^\infty$ is homogeneous.*

**Proof.** Denote by $\varphi(\cdot\,|\,\eta, \gamma)$ the density of $\mathcal{N}_1(\eta, \gamma)$. Then, $y_k^n$'s defined in (11) may be expressed, in the light of (9), as

$$y_1^1 = \frac{1}{q}\frac{\varphi(z_1\,|\,\mu+\delta, 1)}{\varphi(z_1\,|\,\mu, 1)}, \tag{13}$$

and when $n+1 \geqslant 2$,

$$\begin{aligned} y_k^{n+1} &= \frac{1}{q} y_k^n \frac{\varphi(z_{n+1}\,|\,\bar{\rho}(\mu+\delta), 1)}{\varphi(z_{n+1}\,|\,\bar{\rho}\mu, 1)} && (1 \leqslant k \leqslant n), \\ &= \frac{1}{q}\frac{\varphi(z_{n+1}\,|\,\bar{\rho}\mu+\delta, 1)}{\varphi(z_{n+1}\,|\,\bar{\rho}\mu, 1)} && (k = n+1), \\ &= q^{k-n-2} && (k > n+1). \end{aligned} \tag{14}$$

Employing this in (12), we get

$$(1-\pi_{n+1})^{-1} = 1 + \frac{\pi_n}{1-\pi_n}\frac{\varphi(z_{n+1}\,|\,\bar{\rho}(\mu+\delta), 1)}{\varphi(z_{n+1}\,|\,\bar{\rho}\mu, 1)} + \frac{p\varphi(z_{n+1}\,|\,\bar{\rho}\mu+\delta, 1)}{q\varphi(z_{n+1}\,|\,\bar{\rho}\mu, 1)}. \tag{15}$$

Also,

$$(1-\pi_1)^{-1} = 1 + \left(\frac{\pi}{1-\pi} + p\right)\frac{1}{q}\frac{\varphi(z_1\,|\,\bar{\rho}\mu+\delta, 1)}{\varphi(z_1\,|\,\bar{\rho}\mu, 1)}. \tag{16}$$

Let $\mathscr{G}_n$ be the $\sigma$-field induced by $Z(n)$, $n \geqslant 1$. Define $\mathscr{G}_0 = (\Omega, \emptyset)$. Since the transformation of $X(n)$ to $Z(n)$ is 1-1 and Borel measurable, $\mathscr{G}_n = \mathscr{F}_n$ $(\forall_n)$. We are now in a position to infer that $(\Pi_n, \mathscr{F}_n)_0^\infty$ is a transitive sequence, and from the independence of $Z_{n+1}$ and $\mathscr{F}_n$, we get

$$P^\pi(Z_{n+1} \in B\,|\,\mathscr{F}_n) = P^\pi(Z_{n+1} \in B\,|\,\pi_n)$$

for every Borel set $B$ on the state space of $Z$. We may now assert, by virtue of Lemma 2.17 of Shiryaev (1978), that $(\Pi_n, \mathscr{F}_n)_0^\infty$ is Markov. Further, since a fixed function yields $\pi_n$ from $\pi_{n-1}$ (see (15)) for $n \geqslant 2$, $(\Pi_n)_2^\infty$ is homogeneous. $\square$

**Remark.** The recursive expressions for $\pi_n$ in (15) and (16) above are also useful in successive computation of the posterior probability.

### *3.3. Optimal rule in the $\alpha$-unrestricted case*

Let $\Pi_\tau$ be $\mathscr{F}_\tau$-measurable such that $\Pi_\tau = \Pi_n$ on $\{\tau = n\}$, where $\mathscr{F}_\tau = \{A \in \mathscr{F}_\infty\colon A \cap \{\tau = n\} \in \mathscr{F}_n,\ n \in \mathbb{N}\}$, $\mathscr{F}_\infty$ being the $\sigma$-field generated by $(\mathscr{F}_n)$. We may express (4) in terms of $\Pi_n$'s as

$$\inf_{\mathcal{T}} E^{\pi}\left[ w(1-\pi_\tau)+\sum_{k=0}^{\tau-1} \pi_k \right] \quad (\pi>0). \tag{17}$$

This is a stopping problem with 'sampling costs' for the process $(\Pi_n)$: $\Pi_n$ is the 'sampling cost' at the $n$-th stage and on $\{\tau=n\}$ the terminal decision loss is $w(1-\Pi_n)$. Since $(\Pi_n, \mathcal{F}_n)_2^\infty$ has been shown to be a homogeneous Markov sequence, we may derive the optimal rule for the problem specified by (17) in a manner analogous to the independent sequence case (cf. Shiryaev, 1978).

**Theorem 2.** *The rule* $\tau^* = \inf\{n \geqslant 2 \colon \Pi_n > \gamma(w)\}$ *is optimal for the problem specified by*

$$\inf_{\mathcal{T}_2} E^{\pi}\left[ w(1-\Pi_\tau)+\sum_{k=0}^{\tau-1} \Pi_k \right] \tag{18}$$

*where* $\mathcal{T}_2=\{\tau\in\mathcal{T}\colon \tau\geqslant 2\}$.

**Proof.** Define $g(\pi)=w(1-\pi)$ and $Qg(\pi)=\min\{g(\pi), \pi+Tg(\pi)\}$, where $Tg(\pi)=E^{\pi}g(\Pi_1)$. Then, by the discussion in Chapter 2 of Shiryaev (1978), $\lim_N Q^N g(\pi)$ is the infimal 'risk' and

$$\tau^0=\inf\left\{ n\geqslant 2 \colon \lim_{n\to\infty} Q^N g(\pi_n)=g(\pi_n)\right\}$$

would be optimal in $\mathcal{T}_2$ provided that $\tau^0<\infty$ a.s. $(P^{\pi})$. Concavity and continuity of $g(\pi)$ and $\lim_N Q^N g(\pi)$ permit us to assert that a constant $\nu=\nu(w)$, $0<\nu<1$, exists such that

$$\tau^0=\inf\{n\geqslant 2\colon \pi_n\geqslant \nu\},$$

Finiteness of $\tau^0$ may now be established by showing that $\pi_n$ can take values arbitrarily close to 1 for some $n<\infty$. This is evident on application of Lévy's theorem (see Corollary 1 to Lemma 7.4.2 in Chow and Teicher, 1979) to the sequence of r.v.'s

$$Y_n=1(\{\theta\leqslant n\}), \quad n\in\mathbb{N},$$

which gives

$$E^{\pi}(Y_n \mid \mathcal{F}_n)\to E^{\pi}(1(\Omega)\mid \mathcal{F}_\infty)=1.$$

Thus $\tau^0=\tau^*$, the rule optimal for the problem specified by (18). □

**Remark.** Restriction to $\tau_2$ in (18) is due to the nonhomogeneity of $(\Pi_n)$ at $n=1$. Nevertheless, if we consider

$$\inf_{J} E^{\pi_1}\left[ w(1-\Pi_\tau)+\sum_{0}^{\tau-1} \Pi_k \right],$$

where $E^{\pi_1}$ denotes expectation under $P^{\pi_1}$, the probability measure determined by $P^{\pi}$ given $\Pi_1 = \pi_1$, then stopping at $n = 1$ may be included.

### 3.4. *The α-unrestricted case*

Let

$$\mathscr{T}(\alpha) = \{\tau \in \mathscr{T}, \alpha(\tau) \leqslant \alpha\}, \quad 0 < \alpha < 1,$$

predetermined. Suppose that in (17) we limit the search to $\mathscr{T}_2(\alpha)$. Our object would then be to find $\tau_\alpha^*$ such that

$$E^{\pi}(\tau_\alpha^* - \theta)^+ = \inf_{\mathscr{T}_2(\alpha)} E^{\pi}(\tau - \theta)^+. \tag{19}$$

We assert that $\tau_\alpha^*$ is $\tau^*$ of Theorem 2 with $\nu$ determined to satisfy $\alpha(\tau^*) = \alpha$. This assertion does not require a separate proof since the corresponding result in the case of an independent sequence does not rely on independence as such (see Theorem 8, Section 4.4, Shiryaev, 1978). We thus have the following theorem.

**Theorem 3.** $\tau_\alpha^* = \inf\{n \geqslant 2\colon \pi_n \geqslant \nu(\alpha)\}$, *where* $\nu(\alpha)$ *satisfies* $\alpha(\tau_\alpha^*) = \alpha$.

There is no simple way of determining $\nu$ exactly. However, it has the following upper bound which can serve as an approximate value. For every $\tau \in \mathscr{T}_2(\alpha)$,

$$\alpha(\tau) = E^{\pi}(1 - \Pi_\tau) \leqslant 1 - \nu(\alpha) \;\Rightarrow\; \nu(\alpha) \leqslant 1 - \alpha.$$

We consider sequential detection of a change in the regression coefficient in the next section.

## 4. Detecting a change in the regression coefficient when serial correlation is present

Suppose that we are observing a bivariate sequence $(X_n, Y_n)_1^\infty$, and $\theta$ is the unobservable r.v. whose distribution is given by (1). Assume further that *when* $\theta = k$,

$$\begin{aligned} Y_n &= \beta X_n + \varepsilon_n, \quad 1 \leqslant n < k, \\ &= \beta' X_n + \varepsilon_n, \quad n \geqslant k, \end{aligned} \tag{20}$$

where $(\varepsilon_n)$ is a first-order autoregressive process with parameter $\rho$, $|\rho| < 1$, and $X_n$'s are i.i.d. like $\mathcal{N}_1(\mu, 1)$. The problem as stated in (4) of online detection of the instant of change in regression coefficient from $\beta$ to $\beta'$ reduces to the stopping problem for $(\Pi_n)$ stated in (17). Thus we only need to check whether $(\Pi_n, \mathscr{F}_n)$ is Markov.

Box and Tiao (1965) have considered a similar problem. They test for change at a specified point in a time series given one sample up to the hypothesized instant of change and another fixed-size sample from the change point onwards. Our concern is with optimal *online* detection of change point. The transformation

$$Z_1 = Y_1, \qquad Z_n = Y_n - \rho Y_{n-1}, \quad n = 2, 3, \ldots,$$

yields, as shown by Box and Tiao, an independent sequence $(Z_n)$. The conditional distribution of $Z(n|k)$ given $X(n|k) = x(n|k)$ is $\mathcal{N}_n(\mu(x(n|k)), I(n))$ where $\mu(x(n|k))$, defining $x_0 = 0$, is the vector

$$(\beta x_1, \beta(x_2 - \rho x_1), \ldots, \beta(x_{k-1} - \rho x_{k-2}), \beta' x_k - \rho\beta x_{k-1},$$
$$\beta'(x_{k+1} - \rho x_k), \ldots, \beta'(x_n - \rho x_{n-1}))$$

for $k > 1$, and $(\beta' x_1, \beta'(x_2 - \rho x_1), \ldots, \beta'(x_n - \rho x_{n-1}))$, for $k = 0, 1$. Unconditionally, $Z(n|k)$ has the $\mathcal{N}_n(\mu(n|k), I(n)$ distribution with

$$\begin{aligned}
\mu(n|k) &= (\beta\mu, \beta\mu\bar{\rho}, \ldots, \beta\mu\bar{\rho}, \mu(\beta' - \beta\rho), \beta'\mu\bar{\rho}, \ldots, \beta'\mu\bar{\rho}), && k \geq 3,\\
&= (\beta'\mu, \beta'\mu\bar{\rho}, \ldots, \beta'\mu\bar{\rho}), && k = 0, 1,\\
&= (\beta\mu, \mu(\beta' - \beta\rho)), \beta'\mu\bar{\rho}, \ldots, \beta'\mu\bar{\rho}), && k = 2,
\end{aligned}$$

Equation (12) continues to be valid in the present case with the following expressions for $y_k^n$'s defined in (11):

$$\begin{aligned}
y_1^1 &= \frac{1}{q}\,\frac{\varphi(z_1|\beta'\mu, 1)}{\varphi(z_1|\beta\mu, 1)},\\
y_k^{n+1} &= \frac{1}{q} y_k^n \frac{\varphi(z_{n+1}|\beta'\mu\bar{\rho}, 1)}{\varphi(z_{n+1}|\beta\mu\bar{\rho}, 1)}, && 1 \leq k \leq n,\\
&= \frac{1}{q}\,\frac{\varphi(z_{n+1}|\mu(\beta' - \beta\rho), 1)}{\varphi(z_{n+1}|\beta\mu\bar{\rho}, 1)}, && k = n+1,\\
&= q^{k-n-2}, && k > n+1.
\end{aligned}$$

On substitution of these in (12), we get

$$(1 - \pi_1)^{-1} = 1 + \left(\frac{\pi}{1 - \pi} + p\right)\frac{\varphi(z_1|\beta'\mu, 1)}{\varphi(z_1|\beta\mu, 1)},$$

and for $n \geq 1$,

$$(1 - \pi_{n+1})^{-1} = 1 + \frac{\pi_n}{1 - \pi_n}\,\frac{\varphi(z_{n+1}|\beta'\mu\bar{\rho}, 1)}{\varphi(z_{n+1}|\beta\mu\bar{\rho}, 1)} + \frac{p\varphi(z_{n+1}|(\beta' - \beta\rho)\mu, 1)}{\varphi(z_{\eta+1}|\beta\mu\bar{\rho}, 1)}.$$

It is now clear that, by arguments identical to those in Lemma 1, $(\Pi_n)_2^\infty$ is a homogeneous Markov sequence relative to $(\mathcal{F}_n)$, the sequence of histories of $(Y_n)$. Hence the discussion in Sections 3.3 and 3.4 can be carried over to the present case. In particular, Theorems 2 and 3 hold also for the problem in the present section.

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